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INTERIOR AND EXTERIOR RESONANCES

IN ACOUSTIC SCATTERING

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H. Überall, and D. Brill

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I: SPHERICAL TARGETS. In acoustic scattering from elastic objects, resonances features appear in the returned echo at frequencies where the object's eigenfrequencies are located, which are explained by the excitation of interior creeping waves. Corresponding resonance terms may be split off from the total scattering amplitude, leaving behind an apparently nonresonant background amplitude. This is demonstrated here for scatterers of spherical geometry, and

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in the following companion paper also for scatterers or arbitrary geometry, using the T-matrix approach. For the case of near-impenetrable spheres, it is subsequently shown that the background amplitude can be split further into specularly reflected contributions, plus highly attenuated resonance terms which are explained by the excitation of exterior (Franz-type) creeping waves. The singularity structure of the scattering function is shown mathematically, using the R-matrix approach of Nuclear scattering theory, as that of a meromorphic function without any additional "entire function" (as had been postulated by the Singularity Expansion Method).

II: TARGETS OF ARBITRARY SHAPE (T-MATRIX APPROACH). The T-matrix approach, which describes acoustic-wave scattering (or that of other waves) from objects of arbitrary shape and geometry, is here "married" to the resonance scattering theory in order to obtain the complex resonance frequencies of an arbitrarily-shaped target. For the case of near-impenetrable targets, we split the partial-wave scattering amplitude into terms corresponding to internal resonances, plus an apparently non-resonant background amplitude which, however, contains the broad resonances caused by external diffracted (or Franz-type, creeping) waves together with a geometrically-reflected contribution.

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INTERIOR AND EXTERIOR RESONANCES IN ACOUSTIC SCATTERING I: SPHERICAL TARGETS

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In acoustic scattering from elastic objects, resonances features appear in the returned echo at frequencies where the object's eigenfrequencies are located, which are explained by the excitation of <u>interior</u> creeping waves. Corresponding resonance terms may be split off from the total scattering amplitude, leaving behind an apparently nonresonant background amplitude. This is demonstrated here for scatterers of spherical geometry, and in the following companion paper also for scatterers of arbitrary geometry, using the T-matrix approach. For the case of near-impenetrable spheres, it is subsequently shown that the background amplitude can be split further into specularly reflected contributions, plus highly attenuated resonance terms which are explained by the excitation of <u>exterior</u> (Franz-type) creeping waves. The singularity structure of the scattering function is shown mathematically, using the R-matrix approach of Nuclear scattering theory, as that of a meromorphic function <u>without</u> any additional "entire function" (as had been postulated by the Singularity Expansion Method).

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INTRODUCTION

The resonance scattering theory[1] has clarified the role of the resonance features that appear in acoustic-wave scattering (as well as in the scattering of electromagnetic or elastic waves) from fluid or solid objects, by resolving each partial-wave scattering amplitude into contributions of purely resonant form, plus an additional, apparently non-resonant "background" contribution. Physically, the scattering process contains a variety of waves that are generated by the interaction of the incident wave with the target. Some of these are of a geometrical nature, i.e. one has a specularly reflected wave[2], as well as "transmitted" waves[3] that get refracted through the penetrable object. Others take the form of circumferential waves or "surface waves", of which diffractive-type "creeping waves"[2] that propagate around the object externally (i.e., in the ambient medium), jointly with the mentioned reflected-wave contribution, form a background amplitude that appears to be non-resonant when plotted vs. frequency. However, other types of circumferential waves propagate around the object <u>internally</u> (i.e. inside the object itself, with speeds comparable to those of the bulk waves in the material of the object), and it is these latter waves that give rise to the more obvious resonant contributions in the scattering amplitude. Physically, the resonances arise from the phase matching that takes place in the course of the repeated circumnavigations of the surface waves, when the frequency of the incident wave coincides with one of the frequencies of the target object's eigenvibrations.

The external creeping waves also exhibit a phase matching at certain external eigenfrequencies, and hence they become resonant themselves, but these resonances generally are very broad because of high attenuation due to radiation, so that the background amplitude of which they are a part appears non-resonant. In contrast, the internal surface waves are generally very little attenuated, so that they can build up to form quite striking resonance features.

In the following, the separation of the scattering amplitude in terms of these various contributions is carried out in some mathematical detail. For the present discussion, we shall restrict ourselves to the following two limiting cases only:

(a) Nearly-soft objects. This situation is exemplified in acoustics by the case of scattering from an air bubble in water[4], or for elastic-wave scattering by, e.g., a water-filled cavity in aluminum[5].

(b) Nearly-rigid objects. This is exemplified in acoustics by the case of scattering from a water droplet in air[6], or from an aluminum or steel object in water[7].

From the examples quoted, it is seen that the limiting cases (a) and (b) considered here cover quite a large variety of cases of practical interest.

Intermediate cases, such as those of acoustic scattering from a rubber sphere in water, or from a thin, air-filled aluminum shell in water[8], will not be incuded in the present discussion. For the latter case, a phenemenological or "ad-hoc" theory has been devised[8] which introduces an "intermediate background" amplitude, appropriate for objects that are neither nearly-soft not nearly-

rigid. This theory, however, has not been found to be sufficiently general so as to describe, e.g., the case of rubber objects in water.

In the present paper, our discussion of the separation of internal-resonance terms from "non-resonant" background terms in the scattering amplitude will be restricted to objects of separable geometry, and more specifically, to spherical targets. In the following paper[9], a similar separation will be carried out for objects of arbitrary geometry, using the T-matrix theory of acoustic scattering due to Waterman[10].

The resonance frequencies of the internal surface waves, for the two limiting cases considered by us here, can be obtained approximately as follows: for a nearly-soft body, e.g., where the scattering problem closely satisfies the Dirichlet boundary condition, the internal resonance frequencies are approximately those of the case where the interior substance of the object is contained within a rigid enclosure, i.e., closely satisfying the Neumann boundary condition; and vice versa for a nearly-rigid body.

The background amplitude is separated into a geometrical reflection contribution plus Franz-type creeping wave contributions in the standard way, using the Watson transformation. Geometrically transmitted waves are also obtained by a use of that transformation. The resonant as well as background amplitudes are shown to have the singularity structure of a meromorphic function, with details provided by a use of "R-matrix" scattering techniques as developed in Nuclear Physics.

We finally show that if one averages the squared scattering amplitude over frequency, the contributions of the external resonances

drop out and one is left with the background term contribution only. This mean that the averaged cross section equals that of an impenetrable body of the same shape.

I. INTERNAL RESONANCES VS. BACKGROUND

We shall first carry out a separation of the acoustic scattering amplitude into a contribution from the internal resonances, plus an apparently non-resonant background contribution. This is done here for scattering objects of separable (spherical) geometry, while the following paper carries out the same program for scattering objects of arbitrary (non-separable) geometry. Whenever reference to a scatterer of specific composition is made, the case of a gas bubble in a fluid will be chosen for purposes of illustration.

The pressure amplitude of an incident plane wave in a liquid may be written as[4]

$$p_{inc} = \sum_{n=0}^{\infty} (2n+1)\, i^n j_n(kr) P_n(\cos\theta), \tag{1a}$$

where $k = \omega/c$ is the wave number in the liquid; a time factor $\exp(-i\omega t)$ has been suppressed, and a peak amplitude of unity was assumed. For the scattered wave, one has[4] for the case of a spherically-shaped scattering object.

$$p_{sc} = \sum_{n=0}^{\infty} (2n+1)\, i^n T_n h_n^{(1)}(kr) P_n(\cos\theta), \tag{1b}$$

with a partial-wave scattering amplitude (or T-function), T_n. It is customary to write the total pressure amplitude in the form

$$p = p_{inc} + p_{sc}$$
$$= \frac{1}{2}\sum_{n=0}^{\infty}(2n+1)\, i^{n}\left[h_n^{(2)}(kr) + S_n h_n^{(1)}(kr)\right] P_n(\cos\theta), \qquad (1c)$$

where the amplitude S_n of the outgoing ($h_n^{(1)}$) relative to the incoming ($h_n^{(2)}$) spherical partial wave is called the S-function; we have

$$S_n = 2T_n + 1, \qquad T_n = \tfrac{1}{2}(S_n - 1). \qquad (2a)$$

A scattering phase shift δ_n (real for a non-absorbing scatterer, complex for an absorbing scatterer) may be introduced by

$$S_n = \exp(2i\delta_n), \qquad (2b)$$

which leads to

$$T_n = i e^{i\delta_n} \sin\delta_n . \qquad (2c)$$

With the asymptotic form ($r \to \infty$)

$$h_n^{(1)}(kr) \sim (1/kr)\, i^{-n-1} e^{ikr}, \qquad (3a)$$

and introducing the form function

$$f(\theta) = \sum_{n=0}^{\infty} f_n(\theta), \qquad (3b)$$

which consists of a Rayleigh normal-mode series of partial-wave form functions

$$f_n(\theta) = (2/ika)(2n+1)\, T_n\, P_n(\cos\theta), \qquad (3c)$$

we obtain the asymptotic scattering amplitude

$$p_{sc} \sim p_{sc}^{as} = (a/2r)\, e^{ikr} f(\theta), \qquad (3d)$$

expressed in terms of S_n or T_n via Eqs. (3b, c) and (2a).

For acoustic scattering from a solid elastic sphere, it was shown[7] that the S-function may be written in either of two ways, namely as

$$S_n = S_n^{(s)} \frac{F_n - z_n^{(2)}}{F_n - z_n^{(1)}} , \tag{4a}$$

where

$$S_n^{(s)} = - \frac{h_n^{(2)}(x)}{h_n^{(1)}(x)} \equiv e^{2i\xi_n^{(s)}} \tag{4b}$$

is the S-function of a soft sphere; or as

$$S_n = S_n^{(r)} \frac{1/F_n - 1/z_n^{(2)}}{1/F_n - 1/z_n^{(1)}} \tag{4c}$$

where

$$S_n^{(r)} = - \frac{h_n^{(2)\prime}(x)}{h_n^{(1)\prime}(x)} \equiv e^{2i\xi_n^{(r)}} \tag{4d}$$

is the S-function of a rigid sphere. Here, $x = ka$ where a is the radius of the sphere, and

$$z_n^{(i)} = x h_n^{(i)\prime}(x)/h_n^{(i)}(x), \qquad i = 1, 2. \tag{4e}$$

The form of Eq. (4a) is more appropriate for nearly-soft objects, and that of Eq. (4c) for nearly-rigid objects. An example of the former is the case of a gas bubble in a fluid, where one has[4]

$$F_n = \frac{\rho}{\rho_0} \frac{\beta x j_n'(\beta x)}{j_n(\beta x)}, \qquad \beta = c/c_0 \tag{5}$$

where ρ_0 and c_0 are density and sound speed of the object, respectively, and ρ and c the same quantities for the ambient medium. While for an air bubble in water $\beta \sim 5$, the density ratio

is $\rho/\rho_0 \sim 10^3$ so that $F_n \gg 1$ in general. The quantity $1/F_n$ is related to the mechanical impedance of the object[11], and F_n is proportional to ρ/ρ_0 for both fluid and elastic objects. One sees that for $\rho \gg \rho_0$, $S_n \to S_n^{(s)}$ and for $\rho_0 \gg \rho$, $S_n \to S_n^{(r)}$ so that the statement after Eq. (4e) is justified.

The desired separation is now obtained as

$$S_n = S_n^{(s)} + S_n^{(s)\,int}, \qquad S_n^{(int)} = S_n^{(s)} \frac{z_n^{(1)} - z_n^{(2)}}{F_n - z_n^{(1)}} \tag{6a}$$

or

$$T_n = T_n^{(s)} + T_n^{(s)\,int}, \qquad T_n^{(s)\,int} = \frac{1}{2} S_n^{(s)} \frac{z_n^{(1)} - z_n^{(2)}}{F_n - z_n^{(1)}} \equiv \frac{1}{2} S_n^{(s)\,int} \tag{6b}$$

for the nearly-soft case where $T_n^{(s)} = i\exp(i\xi_n^{(s)})\sin\xi_n^{(s)}$, and

$$S_n = S_n^{(r)} + S_n^{(r)\,int}, \qquad S_n^{(int)} = S_n^{(r)} \frac{1/z_n^{(1)} - 1/z_n^{(2)}}{1/F_n - 1/z_n^{(1)}} \tag{7a}$$

or

$$T_n = T_n^{(r)} + T_n^{(r)\,int}, \qquad T_n^{(r)\,int} = \frac{1}{2} S_n^{(r)} \frac{1/z_n^{(1)} - 1/z_n^{(2)}}{1/F_n - 1/z_n^{(1)}} \equiv \frac{1}{2} S_n^{(r)\,int} \tag{7b}$$

for the narly-rigid case, where $T_n^{(r)} = i\exp(i\xi_n^{(r)})\sin\xi_n^{(r)}$. The first term in these expressions represents an apparently non-resonant background amplitude corresponding to scattering from a soft or rigid sphere, respectively, while the second term corresponds to resonant amplitudes. A similar separation is being carried out for the case of objects of arbitrary shape in the following paper, based on Waterman's T-matrix theory[10].

A separation of S_n of T_n into a "non-resonant" intermediate background (e.g. for the case of a shell of intermediate thickness) plus resonances, has been carried out in an ad-hoc fashion[8] which was found applicable for some situations but not for others, such as a rubber sphere in water. The present paper only deals with nearly-impenetrable bodies where the separations of Eqs. (6) and (7) are appropriate.

The resonances in Eqs. (6) and (7) arise from the denominators attaining a minimum value. The eigenvalue equations (characteristic equations)

$$F_n = z_n^{(1)} \quad \text{(nearly-soft)} \qquad (8a)$$

or

$$1/F_n = 1/z_n^{(1)} \quad \text{(nearly-rigid)} \qquad (8b)$$

are solved for complex eigenfrequencies $x = \tilde{x}_{n\ell}^{(s,r)}$ ($\ell = 1,2,\ldots$), respectively, since $z_n^{(1)}$ is complex; for viscoelastic bodies F_n is complex also. Many examples in the literature[1] show the appearance of high, narrow resonances in mechanical systems, suggesting that the imaginary part of the characteristic equations is small in these cases, so that real resonances frequencies

$$x_{n\ell}^{(s,r)} \cong \mathrm{Re}\, \tilde{x}_{n\ell}^{(s,r)} \qquad (9)$$

are approximately given as the solutions of the equations

$$F_n = \mathrm{Re}\, z_n^{(1)} \quad \text{(nearly-soft)} \qquad (10a)$$

or

$$1/F_n = \mathrm{Re}\,(1/z_n^{(1)}) \quad \text{(nearly-rigid)} \qquad (10\text{-}b)$$

The resonances encountered in this section clearly depend on the material parameters of the scattering object contained in F_n, *cf.* Eq. (5), and they disappear from Eqs. (6) for $F_n \to \infty$ (soft-body limit), or from Eqs. (7) for $F_n \to 0$ (rigid-body limit). We may thus refer to them as "internal resonances", and shall show below that they arise from the phase matching of internal circumferential waves which circumnavigate the object underneath its surface.

II. INTERNAL RESONANCE FREQUENCIES AND BOUNDARY CONDITIONS

We shall indicate here the way how for nearly-soft or nearly-rigid bodies, approximate internal resonance frequencies may be obtained by solving appropriate boundary-value problems.

(a) Nearly-Soft Scatterer. The scattering function of this object, Eq. (4a) or (6a), has been obtained by satisfying the appropriate boundary condition of the scattering problem, which is approximately the Dirichlet boundary condition (in the soft limit). How are the resonance frequencies obtained in that approximation? This is illustrated by the example of an air bubble in water, Eq. (5). Here $\rho/\rho_0 \sim 10^3$ so that in general, $F_n = 0(10^3) \gg 1$. Accordingly, the characteristic equation Eq. (8a) is approximately

$$F_n \cong 0, \tag{11a}$$

which by inspection of Eq. (5) is satisfied by

$$j_n'(\beta x) = 0. \qquad (11b)$$

This equation, however, corresponds to the Neumann boundary condition for the internal problem, i.e. the problem of a gas enclosed in a rigid spherical cavity, since the internal field of the bubble is given by[4]

$$p_{int} = \sum_{n=0}^{\infty} (2n+1) i^n j_n(\beta k r) P_n(\cos\theta). \qquad (11c)$$

Equation (11b) represents the characteristic equation for finding the (real) eigenfrequencies of this problem. We may state, therefore, in a general fashion that approximate internal resonance frequencies of a nearly-soft (Dirichlet) scattering object may be found by solving the interior Neumann problem of the same object contained in a rigid enclosure.

This is a further consequence of the fact that at and only at the internal resonance frequencies, the incident field penetrates the scatterer[7].

(b) Nearly-rigid scatterer. The situation here is the reverse from above. The Scattering function of the nearly-rigid object, given by Eq. (4c) or (7a), is obtained by satisfying the boundary condition for the scattering problem of this object, which is approximately the Neumann boundary condition. The near-rigid limit corresponds to $\rho_0 \gg \rho$ so that $F_n \ll 1$. Accordingly, Eq. (8b) leads to the approximate characteristic equation

$$1/F_n \cong 0 \qquad (12a)$$

for this case. Taking the example of a heavy fluid sphere imbedded in a light fluid (such as a water droplet in air[6]), inspection of Eq. (5) shows the characteristic equation to be equivalent to

$$j_n(\beta x) = 0. \tag{12b}$$

Equation (11c) identifies this as corresponding to the <u>Dirichlet boundary condition</u> for the <u>internal problem</u>, i.e. the problem of a fluid sphere in vacuo, and Eq. (12b) represents the characteristic equation for finding the (real) eigenfrequencies of this problem. Again, it may be stated generally that approximate internal resonance frequencies of a nearly-rigid (Neumann) scattering object may be found by solving the characteristic equation of the interior Dirichlet problem of the same object in vacuo.

III. <u>SINGULARITY STRUCTURE OF THE SCATTERING FUNCTION (R-MATRIX THEORY)</u>

As discussed above, the expressions $S_n^{(s,r)\,int}$ or $T_n^{(s,r)\,int}$ of Eqs. (6) and (7) contain the internal resonances of the scattering object. The corresponding complex resonance frequencies $\tilde{x}_{n\ell}$ correspond to poles of $S_n^{(s,r)\,int}$ or $T_n^{(s,r)\,int}$ in a complex frequency plane (or x-plane), which lie close to the real axis (i.e., $\tilde{x}_{n\ell}^{(s,r)}$ having a small imaginary part) for the two limiting cases of nearly-soft or nearly-rigid scatterers considered here. The explicit singularity structure of $S_n^{(s,r)\,int}$ or $T_n^{(s,r)\,int}$ can be obtained by making use of Wigner's R-Matrix theory[12] as developed in nuclear physics[13]. We separate the quantities

$$z_n^{(i)}/x \equiv h_n^{(i)\prime}(x)/h_n^{(i)}(x) = \sigma_n^{(s)}(x) \pm i\pi_n^{(s)}(x), \qquad i=1,2 \quad \text{(13a)}$$

or

$$x/z_n^{(i)} \equiv h_n^{(i)}(x)/h_n^{(i)\prime}(x) \equiv \sigma_n^{(r)}(x) \pm i\pi_n^{(r)}(x), \qquad i=1,2 \quad \text{(13b)}$$

into their real (σ_n) and imaginary parts (π_n), and introduce the R-functions

$$R_n^{(s)}(x) \equiv x/F_n(x) = \sum_{\ell=1}^{\infty} \frac{\gamma_{n\ell}^{(s)\,2}}{\hat{x}_{n\ell}^{(s)} - x} \quad , \qquad \text{(13c)}$$

or

$$R_n^{(r)}(x) \equiv 1/R_n^{(s)}(x) \equiv F_n(x)/x = \sum_{\ell=1}^{\infty} \frac{\gamma_{n\ell}^{(r)\,2}}{\hat{x}_{n\ell}^{(r)} - x} \quad , \qquad \text{(13d)}$$

shown by Wigner to be represented[12,13] by the meromorphic functions of Eqs. (13c,d) with poles at the locations $x = \hat{x}_{n\ell}^{(s,r)}$ (which are real for scatterers of non-absorbing material). For a fluid sphere, one has[12], e.g. (with $\ell \rightarrow k = -\infty \ldots \infty$):

$$\hat{x}_{nk}^{(s)} = z'_{nk}/\beta, \qquad \text{(14a)}$$

$$\gamma_{nk}^{(s)\,2} = -(\rho_0/\rho\beta^2)\, j_n(z'_{nk})/j''_n(z'_{nk}), \qquad \text{(14b)}$$

where z'_{nk} are the (real) zeros of $j'_n(z)$. Equations (13) inserted in Eqs. (6) and (7) lead to the following expression:

$$S_n^{(s,r)\,int} \equiv 2T_n^{(s,r)\,int} = S_n^{(s,r)} \frac{2i\pi_n^{(s,r)}R_n^{(s,r)}}{1-(\sigma_n^{(s,r)}+i\pi_n^{(s,r)})R_n^{(s,r)}} \quad , \qquad \text{(15)}$$

which together with the meromorphic expression of Eq. (13c,d) for $R_n^{(s,r)}$ clearly exhibits the singularity structure of the scattering functions, namely, as being represented in terms of

meromorphic functions $\mathcal{R}_n^{(s,r)}(x)$.

It may now be argued that $S_n^{(s,r)\,int}$ are themselves meromorphic functions, and this may most easily be demonstrated for the case that the resonances are fairly narrow and well-separated from each other (which is often the case for acoustic resonances[1]). The so-called "one-level approximation" thus considers one pole at a time in Eq . (15), with the result

$$S_n^{(s,r)\,int} \cong \sum_{\ell} S_n^{(s,r)}(\hat{x}_{n\ell}^{(s,r)}) \frac{-2i\pi_n^{(s,r)}(\hat{x}_{n\ell}^{(s,r)})\gamma_{n\ell}^{(s,r)\,2}}{x - x_{n\ell}^{(s,r)} + \frac{1}{2} i \Gamma_{n\ell}^{(s,r)}} \tag{16a}$$

where

$$x_{n\ell}^{(s,r)} = \hat{x}_{n\ell}^{(s,r)} - \sigma_n^{(s,r)}(\hat{x}_{n\ell}^{(s,r)})\gamma_{n\ell}^{(s,r)\,2} \tag{16b}$$

is the real resonance frequency, shifted from the poles of the R_n function by the amount $-\sigma_n \gamma_{n\ell}^2$, and where

$$\Gamma_{n\ell}^{(s,r)} = 2\pi_n^{(s,r)}(\hat{x}_{n\ell}^{(s,r)})\gamma_{n\ell}^{(s,r)\,2} \tag{16c}$$

is the resonance width which is usually small for the external resonances[1]. Accordingly, the complex pole positions of $S_n^{(s,r)\,int}$ or $T_n^{(s,r)\,int}$ are

$$\tilde{x}_{n\ell}^{(s,r)} = x_{n\ell}^{(s,r)} - \frac{1}{2} i \Gamma_{n\ell}^{(s,r)}, \tag{16d}$$

usually lying close to the real axis of the complex x-plane.

In this way, the portions $S_n^{(s,r)int}$ of the S-function have been shown to be meromorphic functions of frequency, with poles located at complex positions in the complex frequency plane. In the theory of radar scattering, the existence and the practical importance for purposes of target identification, of these poles has been pointed out in the framework of the so-called[14] "Singularity Expansion Method" (SEM). In that theory, the additional existence of an entire function in the scattering amplitude has also been postulated. Looking at the total scattering amplitudes Eqs. (6a) and (7a), the question arises whether the impenetrable-target background contributions $S_n^{(s,r)}$ which are of apparent non-resonant nature, might represent this entire function. This has to be answered in the negative, since in Section V it will be shown that $S_n^{(s,r)}$ contain resonances caused by the "external" surface waves (or "creeping waves"), and hence are meromorphic functions themselves. Their poles, in contrast to those of the internal resonances, however, have large imaginary parts and lie far from the real axis, so that the corresponding external resonances are very broad and often not distinguishable as such, so that the background $S_n^{(s,r)}$ may appear to be non-resonant. The absence of an entire function in the scattering amplitudes of several radar scattering problems of interest has also been noted by Pearson[15].

IV INTERNAL SURFACE WAVES

The connection between the internal resonances, and internal surface waves that encircle the scattering object beneath its surface, has been pointed out prviously[4,9]. To summarize briefly, one may transform the complex-frequency-plane poles $\tilde{x}_{n\ell} \equiv x_{n\ell} - \frac{1}{2} i \Gamma_{n\ell}$ (suppressing indices s,r) of the S_n^{int} functions into poles in the complex mode-number plane. The corresponding interpretation of the mode number n as a complex variable, and the following procedure for such a pole tranformation, are completely equivalent to the formalism of the Watson transformation as carried out in Section V, but the present procedure constitutes a simplified version thereof. We Taylor-expand

$$x_{n\ell} \cong x_{n_\ell \ell} + (n - n_\ell)\, x'_{n_\ell \ell} + \cdots \tag{17a}$$

about the value n_ℓ chosen such that

$$x_{n_\ell \ell} = x, \tag{17b}$$

x being the given frequency of the acoustic field. Usually, $x'_{n_\ell \ell} > 0$ since the resonance frequencies are generally found to increase with mode number. The S_n^{int} function of Eq. (16a) then becomes

$$S_n^{(s,r)\,int} \cong \sum_\ell \frac{S_n^{(s,r)}}{x'_{n_\ell \ell}} \frac{2 i \pi_n \gamma_{n\ell}^2}{n - n_\ell - \frac{1}{2} i \tilde{\Gamma}_{n\ell}}, \tag{18a}$$

with resonance widths in the n-variable given by

$$\tilde{\Gamma}_{n\ell} = \Gamma_{n\ell} / x'_{n_\ell \ell} > 0 ; \qquad (18b)$$

it possesses poles in the complex mode-number plane ("Regge poles"), located at positions $\tilde{n}_\ell$ where

$$\tilde{n}_\ell = n_\ell + \tfrac{1}{2} i \tilde{\Gamma}_{n\ell} . \qquad (18c)$$

The Watson transformation now involves an evaluation of the scattering amplitude in terms of the residues at these poles. The physical meaning of the amplitudes is then provided by inspection of the asymptotic forms for the $P_n(\cos\theta)$ functions entering in p_{sc} of Eq. (1b), evaluated at the Regge pole $n = \tilde{n}_\ell$ of Eq. (18c):

$$P_{\tilde{n}_\ell}(\cos\theta) \cong \{2/\pi(\tilde{n}_\ell + \tfrac{1}{2})\sin\theta\}^{1/2} \, \tfrac{1}{2}\sum_{\varepsilon=\pm 1} e^{i\varepsilon(\tilde{n}_\ell+\frac{1}{2})\theta - i\varepsilon\pi/4} . \qquad (19)$$

This factor represents a manifold (labeled by ℓ) of (internal!) "tidal waves" which engulf the spherical objects at all meridians, converging at the north pole ($\theta = 0$), then surging back to converge again at the south pole ($\theta = \pi$), and so forth. At each of the convergence points, a phase loss of $\pi/2$ (i.e., a quarter wavelength) is suffered by these surface waves[8]. The quantity $(\tilde{n}_\ell + \frac{1}{2})$ is the complex propagation constant of the surface waves, whose phase velocities accordingly are given by

$$c_\ell(x) = \{ x/(n_\ell + \tfrac{1}{2}) \} c ; \qquad (20a)$$

when plotted as a function of frequency x, Eq. (20a) gives the dispersion curve of the ℓth surface wave. In Fig. 1, we present the dispersion curves of the surface waves ℓ=1 through 14 as a function of x≡ka relative to the sound velocity in water, c, for an air-filled bubble in water. Note that the curves approach asymptotically the value c_o/c; this agrees with their identification as internal surface waves that propagate in the air filling the bubble.

The attenuation of the surface wave amplitudes, written as $\exp(-\theta/\theta_\ell)$, is given by

$$\theta_\ell = 2 / \tilde{\Gamma}_{n_\ell \ell} , \tag{20b}$$

and the wavelength of the ℓth surface wave is

$$\lambda_\ell (x) = 2\pi a / (n_\ell + \tfrac{1}{2}) . \tag{20c}$$

The Regge pole positions $\tilde{n}_\ell$ move along "trajectories" in the complex mode number plane as the frequency is varied. At the resonance frequency, their real part n_ℓ coincides with the integer n producing a resonance in the scattering amplitude [see Eq. (18a)]; Eq. (20c) shows that here, n + ½ wavelengths span the circumference of the spherical scatterer. Taken together with the two quarter-wave phase shifts at the north and south poles, we see that a phase match occurs in the surface waves upon their repeated circumnavigations, which may both be considered as a mathematical condition for a resonance to occur, and as a physical cause of it.

We have shown here how "internal" resonances are connected with the physical phenomenon of internal surface waves. In the following section, a similar connection will be established for the "external" resonances, and external surface waves ("creeping waves").

V. BACKGROUND RESONANCES AND EXTERNAL SURFACE WAVES ("CREEPING WAVES")

We shall start here with a consideration of the asymptotic scattering amplitude for a soft or a rigid sphere, Eq. (3d):

$$p_{sc}^{(s,r)} = \frac{e^{ikr}}{2ikr} \sum_{n=0}^{\infty} (2n+1)\left(S_n^{(s,r)} - 1\right) P_n(\cos\theta), \tag{21a}$$

with $S_n^{(s,r)}$ given by Eqs. (4b) or (4d), respectively. The standard Watson transformation consists in rewriting the sum in Eq. (21a) as the contour integral[16]

$$p_{sc}^{(s)} = \frac{e^{ikr}}{2kr} \oint_C \lambda \left\{ \frac{h^{(2)}_{\lambda-\frac{1}{2}}(x)}{h^{(1)}_{\lambda-\frac{1}{2}}(x)} + 1 \right\} \frac{P_{\lambda-\frac{1}{2}}(-\cos\theta)}{\cos\pi\lambda}\, d\lambda \tag{21b}$$

(for a soft sphere), where the contour C tightly surrounds the positive real axis in the complex λ - plane in the clockwise sense, passing to the right of the origin. For the rigid sphere,

$h^{(i)}_{\lambda-\frac{1}{2}}(x)$ gets replaced by $h^{(i)\prime}_{\lambda-\frac{1}{2}}(x)$, cf Eqs. (4b) and (4d). One may perform the decomposition

$$P_{\lambda-\frac{1}{2}}(-\cos\theta) = e^{-i\pi(\lambda-\frac{1}{2})} P_{\lambda-\frac{1}{2}}(\cos\theta) - 2i\cos\pi\lambda\, Q^{(-)}_{\lambda-\frac{1}{2}}(\cos\theta), \tag{22a}$$

where we intoduced the functions

$$Q_\ell^{(\pm)}(\cos\Theta) = \frac{1}{2}\left\{P_\ell(\cos\theta) \mp (2i/\pi)\, Q_\ell(\cos\theta)\right\}, \tag{22b}$$

defined in terms of the Legendre functions of second kind, $Q_\ell(\cos\theta)$. Using Eq. (22a) in Eq. (21b), one may split $p_{sc}^{(s,r)}$ into two parts

$$p_{sc}^{(s,r)} = p_{cr}^{(s,r)} + p_g^{(s,r)} \tag{23}$$

according to the two terms in Eq. (22a). The second, or "geometric" term, e.g.

$$p_g^{(s)} = \frac{e^{ikr}}{ikr} \oint_C \lambda \left\{ \frac{h^{(2)}_{\lambda-\frac{1}{2}}(x)}{h^{(1)}_{\lambda-\frac{1}{2}}(x)} + 1 \right\} Q^{(-)}_{\lambda-\frac{1}{2}}(\cos\theta)\, d\lambda \tag{24a}$$

no longer has the poles on the real λ axis which stem from the zeros of $\cos\pi\lambda$, and it is customarily evaluated by the method of steepest descent, with the result[16]

$$p_g^{(s,r)} = \mp \frac{a\, e^{ikr}}{4r} \times e^{-2x\cos\frac{1}{2}\theta} \left\{ 1 + \frac{1}{2x\cos^3\frac{1}{2}\theta} + \cdots \right\}. \tag{24b}$$

This expression may be interpreted as the amplitude of a wave geometrically reflected from the sphere. The first term of Eq. (23), e.g.

$$p_{cr}^{(s)} = \frac{e^{ikr}}{2kr} \oint_C \lambda\, d\lambda \left\{ \frac{h^{(2)}_{\lambda-\frac{1}{2}}(x)}{h^{(1)}_{\lambda-\frac{1}{2}}(x)} + 1 \right\} \frac{e^{-i\pi(\lambda-\frac{1}{2})}}{\cos\pi\lambda} P_{\lambda-\frac{1}{2}}(\cos\theta), \tag{25a}$$

may be evaluated as $2\pi i$ times the sum of residues over the poles of the integrand, the latter being given by

$$\lambda = \lambda_\ell^{(s,r)} \equiv \tfrac{1}{2} + \nu_\ell^{(s,r)}(x), \tag{25b}$$

where $\nu_\ell^{(s,r)}(x)$ are found as the solutions of

$$h_\nu^{(1)}(x) = 0, \qquad \nu = \nu_\ell^{(s)}(x) \text{ (soft sphere)} \tag{25c}$$

or

$$h_\nu^{(1)\prime}(x) = 0, \qquad \nu = \nu_\ell^{(r)}(x)\,(\text{rigid sphere}) \tag{25d}$$

Here, $\ell = 1,2,3\ldots$ labels the manifold of solutions of Eqs.(25c,d). The result of this residue evaluation is, e.g.,

$$p_{cr}^{(s)} = \pi \frac{e^{ikr}}{2ikr} \sum_{\ell=1}^{\infty} (2\nu_\ell + 1) \frac{h_{\nu_\ell}^{(2)}(x)}{\dot{h}_{\nu_\ell}^{(1)}(x)} \frac{e^{-i\pi\nu_\ell}}{\sin\pi\nu_\ell} P_{\nu_\ell}(\cos\theta) \tag{26}$$

(suppressing an index s on ν_ℓ), where $\partial h_\nu^{(1)}(x)/\partial\nu = \dot{h}_\nu^{(1)}(x)$. Franz[2] has obtained $\nu^{(s,r)}(x)$ in the form, e.g.

$$\nu_\ell^{(s)}(x) = -\frac{1}{2} + x + \left(\frac{x}{6}\right)^{1/3} e^{i\pi/3} q_\ell - \left(\frac{6}{x}\right)^{1/3} e^{-i\pi/3}\, \frac{q_\ell^2}{180} + \cdots \tag{27}$$

where q_ℓ are the zeros of the Airy function (q_1 =3.372134, q_2= 5.895843, q_3 = 7.962025...); higher terms in the expansion of Eq. (27) were obtained by Franz and Galle[17]. Together with the asymptotic form of P_{ν_ℓ}, Eq. (19), one has the result that the terms in Eq. (26) represent circumferential waves with phase velocities, e.g. for the soft sphere:

$$\frac{c_\ell^{(s)}}{c} = \frac{x}{\frac{1}{2}+\nu_\ell^{(s)}} = \left\{1 + \tfrac{1}{2}6^{-1/3}x^{-2/3}q_\ell - 6^{4/3}x^{-4/3}q_\ell^2/360 + \cdots\right\}^{-1}, \tag{28a}$$

and attenuation angles

$$\theta_\ell^{(s)} = 2\left\{(x/6)^{1/3}\, 3^{1/2} q_\ell + (6/x)^{1/3}\, 3^{1/2} q_\ell^2/180 + \cdots\right\}^{-1}. \tag{28b}$$

In Fig. 2, we present the dispersion curves $c_\ell(x)/c$ for the first five surface waves on a soft sphere, the latter being close to the case of an air bubble in water (a), and the corresponding attenuation angles $\theta_\ell(x)$ as a function of x (b). In Figs. 3(a) and 3(b), corresponding quantities are shown for the rigid sphere, being close to the case of a water droplet in air, or to a metal sphere in water. The first creeping wave on the rigid sphere is seen to be much less attenuated than all the others.

The fact that soft and rigid spheres are impenetrable, and that the dispersion curves in Figs. 2 and 3 approach the value 1 as $x \to \infty$, so that $c_\ell \to c$ (the sound speed in the external medium), indicates that the circumferential waves contained in the amplitudes $S_n^{(s,r)}$ constitute <u>external</u> surface waves, also known as "creeping waves".

The "external" scattering amplitude of Eq. (26) can be shown to be a meromorphic function of x, since it contains poles at those values of x where $\sin \pi\nu_\ell(x) = 0$, i.e. where $\nu_\ell(x) = n$. We may expand the denominator of Eq. (26) around these poles, with the result

$$p_{cr}^{(s)} = \frac{a}{2ir} e^{ikr} \sum_{\ell=1}^{\infty} \sum_{n=0}^{\infty} \frac{2n+1}{\hat{x}_{n\ell}} \frac{h_n^{(2)}(\hat{x}_{n\ell})}{\dot{h}_n^{(1)}(\hat{x}_{n\ell})} \frac{P_n(\cos\theta)}{\dot{\nu}_\ell(\hat{x}_{n\ell})} \frac{1}{x-\hat{x}_{n\ell}} \tag{29}$$

where $\dot{\nu}_\ell(x) = d\,\nu_\ell(x)/dx$, and where the complex zeros in x of $h_n^{(1)}(x)$ were dented by $\hat{x}_{n\ell}$. These zeros are well-known[14], and all have large imaginary parts. Equation (29) thus shows that $p_{cr}^{(s,r)}$ contains external resonances that are related to the creeping waves in the same way in which the internal resonances of Section III were related to the internal surface waves of Section IV; due to their large imaginary parts, however, the external creeping waves are highly attenuated, see Fig. 2(b) and 3(b).

The resonance structure of $S_n^{(s,r)}$ which was mentioned at the end of Section III, may here be discussed more extensively. Taking $S_n^{(s)}$ of Eq. (4b) as an example, we see that its poles are generated by the roots $\hat{x}_{n\ell}$ of $h_n^{(1)}(x) = 0$, or what amounts to the same, by the roots of $\nu_\ell(x) = n$. If we expand

$$h_n^{(1)}(x) \cong (x-\hat{x}_{n\ell})\, h_n^{(1)\prime}(\hat{x}_{n\ell}), \tag{30a}$$

then we see that $S_n^{(s)}$ contains the pole series

$$S_n^{(s)}(x) \sim -\sum_{\ell=1}^{\infty} \frac{h_n^{(2)}(\hat{x}_{n\ell})}{h_n^{(1)\prime}(\hat{x}_{n\ell})} \frac{1}{x-\hat{x}_{n\ell}} \quad . \tag{30b}$$

Thus, both constituents in S_n of Eqs. (6) and (7), i.e. $S_n^{(s,r)}$ and $S_n^{(s,r)int}$, are meromporphic and no further entire function is present, as mentioned at the end of Section III.

It may be asked whether the meromorphic form of Eq. (29) could not have been derived directly from Eq. (21a), without use of the Watson transformation, simply by the expansion of Eq. (30a). If this is done, one obtains

$$p_{sc}^{s} \to \frac{ia}{2r} e^{ikr} \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} \frac{2n+1}{\hat{x}_{n\ell}} \frac{h_n^{(2)}(\hat{x}_{n\ell})}{h_n^{(1)\prime}(\hat{x}_{n\ell})} \frac{P_n(\cos\theta)}{x-\hat{x}_{n\ell}} \quad , \tag{31a}$$

which agrees with Eq. (29) since by use of the chain rule of differentiation,

$$\dot{h}_n^{(1)}(\hat{x}_{n\ell})\,\dot{\nu}_\ell(\hat{x}_{n\ell}) = -\,h_n^{(1)\prime}(\hat{x}_{n\ell}), \tag{31b}$$

the denominator of Eq.(29) may be simplified and this equation becomes

$$p_{cr}^{(s)} = \frac{ia}{2r} e^{ikr} \sum_{\ell=1}^{\infty}\sum_{n=0}^{\infty} \frac{2n+1}{\hat{x}_{n\ell}} \frac{h_n^{(2)}(\hat{x}_{n\ell})}{h_n^{(1)\prime}(\hat{x}_{n\ell})} \frac{P_n(\cos\theta)}{x-\hat{x}_{n\ell}} \quad . \tag{32}$$

If, however, we had used Eq. (31a) as it stands, the geometrical contribution of Eq.(24b) would have been lost, so that the use

of the Watson transformation constitutes the correct mathematical procedure which leads to both geometrically reflected and external creeping waves in the "background" amplitude $p_{sc}^{(s,r)}$.

As a final note, we remark that for a nearly-soft sphere, it has thus been shown that the real parts of the (slightly complex) internal resonance frequencies are approximately determined as the roots of $j_n'(\beta x) = 0$, Eq. (11b), and the (thoroughly complex) external resonance frequencies as the roots of $h_n^{(1)}(x) = 0$, see above. Conversely, for a nearly-rigid sphere, the real parts of the internal resonance frequencies are approximately the roots of $j_n(\beta x) = 0$, Eq.(12b), and the external resonance frequencies the roots of $h_n^{(1)'}(x) = 0$. This statement should be generalized to nearly-soft or nearly-rigid scatterers of general shapes.

VI. TRANSMITTED GEOMETRICAL WAVES

Although the scattering objects considered here are close to soft or to rigid, they nevertheless possess a degree of penetrability which manifests itself in the existence of the internal resonances and the corresponding internal surface waves. As to "geometrical waves", only an expression, Eq. (24b), that corresponds to an externally reflected contribution, has been identified so far. It is well known, however, that transmitted geometrical waves which orginate due to refraction, and which may also undergo any number of total internal reflections, are present for penetrable scatterers.

They have been analyzed for a fluid cylinder imbedded in another fluid, e.g., by Brill and Überall[18]. We shall, for the sake of completeness, outline their derivation for spherical scatterers, and for simplicity for a fluid sphere imbedded in another fluid.

We make use of Eqs. (4a), (4b) and (4e), together with the expression for F_n of Eq. (5) which specifically refers to the fluid sphere. One may then expand[18] ("Debye expansion"):

$$S_n = S_n^{(3)} \left\{ R_{12} - \frac{h_n^{(1)}(\beta x)}{h_n^{(2)}(\beta x)} T_{12} T_{21} \sum_{k=1}^{\infty} \left(\frac{h_n^{(1)}(\beta x)}{h_n^{(2)}(\beta x)} R_{21} \right)^{k-1} \right\}; \tag{33a}$$

this contains an external reflection coefficient

$$R_{12} = \left\{ \frac{h_n^{(2)\prime}(x)}{h_n^{(2)}(x)} - N \frac{h_n^{(2)\prime}(\beta x)}{h_n^{(2)}(\beta x)} \right\} U^{-1}, \tag{33b}$$

and an internal one,

$$R_{21} = - \left\{ \frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)} - N \frac{h_n^{(1)\prime}(\beta x)}{h_n^{(1)}(\beta x)} \right\} U^{-1} \tag{33c}$$

where

$$U = \frac{h_n^{(1)\prime}(x)}{h_n^{(1)}(x)} - N\,\frac{h_n^{(2)\prime}(\beta x)}{h_n^{(2)}(\beta x)} \;, \qquad (33d)$$

and also two transission coefficients which are:

$$T_{12} = 1 - R_{12} \qquad (33e)$$

for transmission into the sphere, and

$$T_{21} = 1 + R_{21} \;, \qquad (33f)$$

for transmission out of the sphere; further,

$$N = \beta\rho/\rho_0 \,. \qquad (33g)$$

In the limit of a soft sphere, we have $N \gg 1$ so that $R_{12} \to 1$, $T_{12} \to 0$ and thus, $S_n \to S_n^{(s)}$ as expected. For a rigid sphere, $N \ll 1$ so that $R_{21} \to -1$, $T_{21} \to 0$ and thus, $S_n \to S_n^{(r)}$ as expected. If the sphere is not completely soft or rigid (i.e., the case considered here), Eqs. (33) contain the correct effects of penetrability.

In the preceding analysis, we have, e.g. for the soft-sphere case, decomposed S_n into $S_n^{(s)int}$ and $S_n^{(s)}$, and have subsequently applied the Watson transformation to both terms: to $S_n^{(s)int}$ from Eqs. (17a) to (19), and to $S_n^{(s)}$ in Eq. (21b). In the latter case, a geometrical term Eq.(24a) was split off via the use of Eq. (22a).

However, if we had applied the Watson transformation to S_n as a whole before ever decomposing it into $S_n^{(s)int}$ and $S_n^{(s)}$, and then split the Watson integral into two parts by using Eq. (22a), the term containing $Q^{(-)}_{\lambda-\frac{1}{2}}$ would then be interpreted as a geometrical contribution containing both reflected and transmitted waves[18]; and as a matter of fact, when using the expansion of Eq. (33a) for this geometrical contribution, and evaluating the contour integral by the method of steepest descent as after Eq. (24a), the term resulting from R_{12} in Eq. (33a) leads in the limit of $N \gg 1$ to the expression of Eq. (24b) for the amplitude reflected from a soft sphere, while the contribution from the sum over k in Eq. (33a) furnishes the waves refracted into the sphere (T_{12}) and emerging from it again (T_{21}) after having undergone k total internal reflections (R_{21}) inside the sphere - in other words, the transmitted geometrical waves. The latter will not be studied in any more detail here, since they have been analyzed for the similar case of the fluid cylinder earlier[18].

VII. FREQUENCY AVERAGE OVER RESONANCES

A final remark will concern here the observation[20] that the usually narrow, non-overlapping internal resonances when averaged over frequency, disappear from the scattering cross section so that the expression for the latter depends on the contribution of the "non-resonant" background only. This will be shown here for the spherical case, and in the following paper for the general case of an arbitrarily-shaped scatterer.

The differential cross section for acoustic scattering is given by[8]

$$\frac{d\sigma}{d\Omega} = \frac{a^2}{4}\left|\sum_n f_n(\theta)\right|^2, \tag{34a}$$

or, using Eq. (3c), by

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\sum_{nn'}(2n+1)(2n'+1)T_n^* T_{n'} P_n(\cos\theta) P_{n'}(\cos\theta). \tag{34b}$$

From Eqs. (6b) and (16a), and suppressing unneeded indices (s,r), we may write for T_n:

$$T_n = \sum_\ell \frac{N_{n\ell}}{x - x_{n\ell} + \frac{1}{2}i\Gamma_{n\ell}} + T_n^{(s,r)} \tag{35a}$$

where

$$N_{n\ell} = -\frac{1}{2}i\, S_n^{(s,r)}\, \Gamma_{n\ell}. \tag{35b}$$

Inserting Eq. (35a) in Eq. (34b), one obtains

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\sum_{nn'}(2n+1)(2n'+1)\left\{\sum_{\ell\ell'} \frac{N_{n\ell}^* N_{n'\ell'}}{(x - x_{n\ell} - \frac{i}{2}\Gamma_{n\ell})(x - x_{n'\ell'} + \frac{i}{2}\Gamma_{n'\ell'})}\right.$$

$$\left. + 2\,\mathrm{Re}\,\frac{N_{n\ell}^* T_{n'}^{(s,r)}}{x - x_{n\ell} - \frac{1}{2}i\Gamma_{n\ell}} + T_n^{(s,r)*} T_{n'}^{(s,r)}\right\} \tag{36a}$$

$$\cdot P_n(\cos\theta) P_{n'}(\cos\theta).$$

If we now average over frequencies, the first two terms in this expression integrate to zero, and we are left with

$$\left\langle \frac{d\sigma}{d\Omega} \right\rangle = \frac{1}{k^2} \left| \sum_n (2n+1) T_n^{(s,r)} P_n(\cos\theta) \right|^2 , \qquad (36b)$$

which is the cross section for scattering from a rigid or soft sphere.

VIII. CONCLUSION

This study has attempted to present a comprehensive physical description of the phenomena that take place during the scattering of waves from a penetrable obstacle. These turned out to fall into two categories, namely (a) geometric waves (reflected and transmitted), and (b) surface waves (extended and internal). For tne limiting case of an impenetrable scatterer, the waves reduce to (a) reflected, and (b) external surface waves. Although our analysis has been limited to the cases of almost-soft or almost-rigid obstacles as exemplified by spheres, the results are believed to hold (with corresponding modifications) for general smooth penetrable obstacles. The limitation to spherical scatterers will be removed in the following paper.

It has been shown that the discussion of the surface waves may equally be carried out in terms of resonances of the scatterer

(external and internal), since the relation between the surface waves and the resonances has been established, and the physical origin of the resonances has been identified in terms of the phase matching of surface waves.

An additional item still to be investigated is the possibility of resonances occurring in the multiply-internally-reflected refracted waves[21] ("rainbow" and "glory" effects). Apart from this, we believe the present discussion to provide a substantially complete physical picture of the scattering process as it takes place on a penetrable target.

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Figure Captions

Fig. 1. Dispersion curves for the internal circumferential waves of an air bubble in water. The phase velocity $c_\ell(x)$ of the ℓth surface wave, normalized by the sound speed c in water, is plotted vs. $x \equiv \omega a/c$. The curves approach asymptotically the value c_o/c where c_o is the speed of sound in air.

Fig. 2. Dispersion curves for the normalized phase velocities (a), and attenuation angles θ_ℓ (b), for the creeping waves (labeled by ℓ) on a soft sphere, plotted vs. $x = ka$.

Fig. 3. Same as Fig. 2, for a rigid sphere.

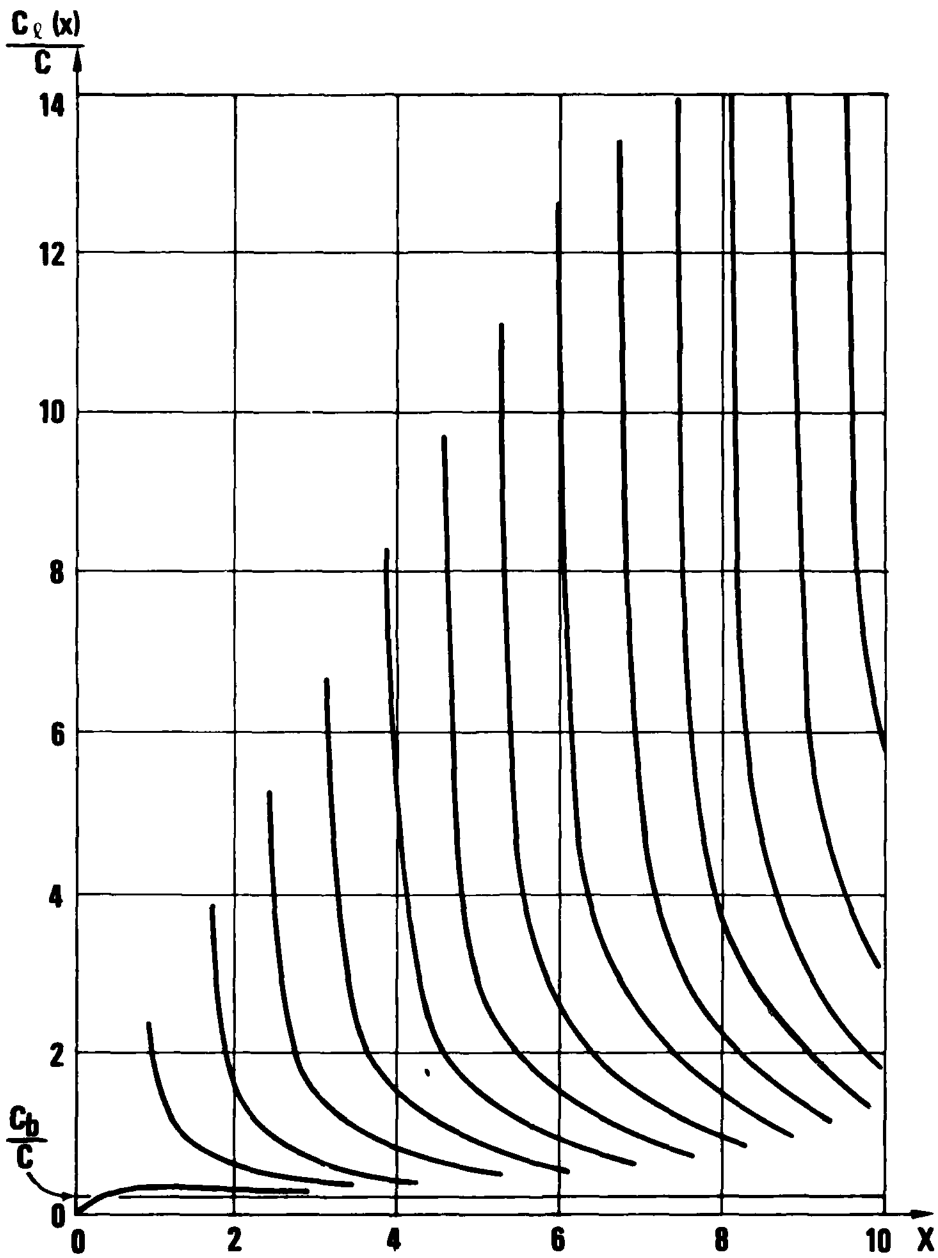

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Fig. 1

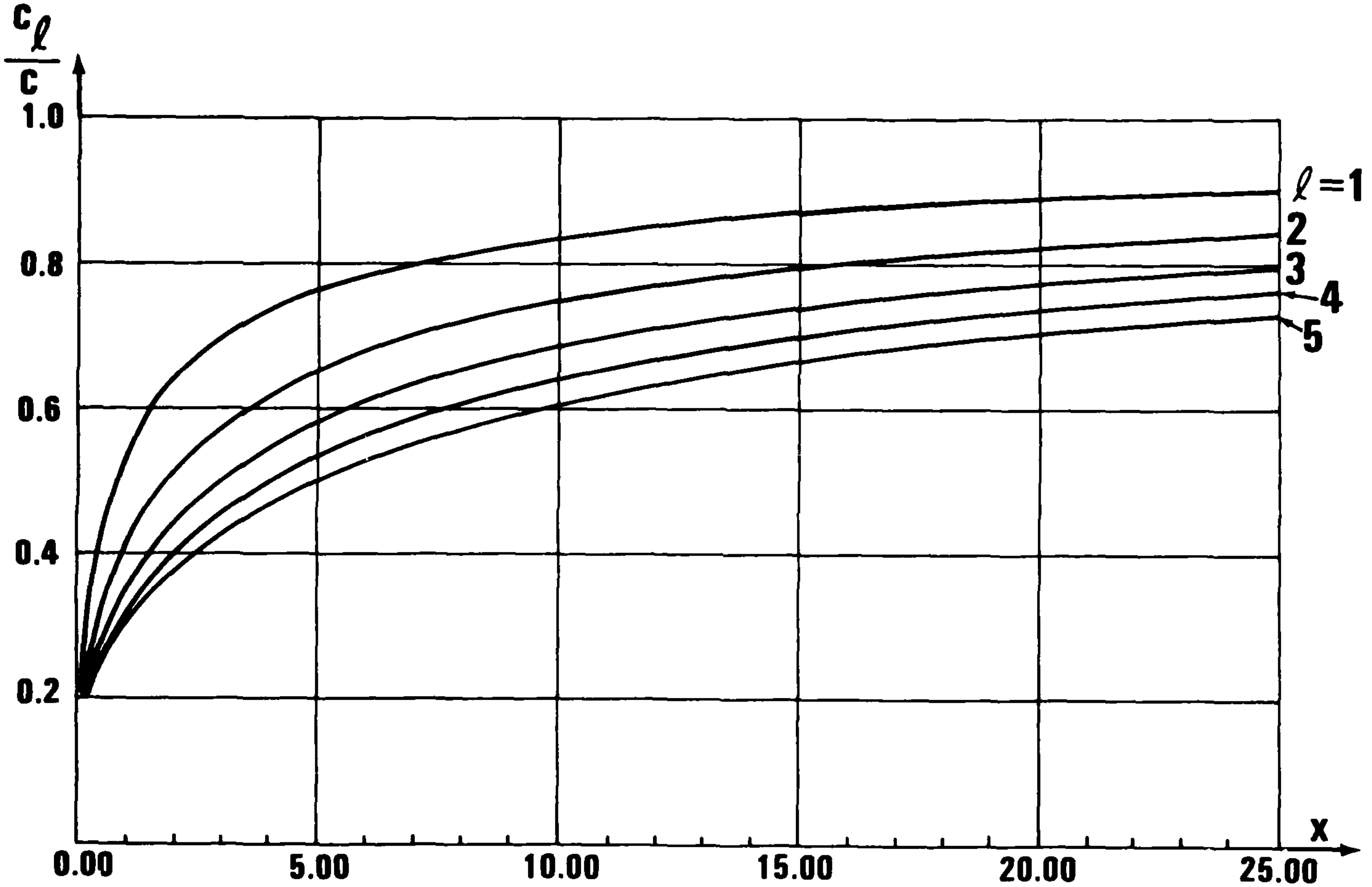

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Fig. 2a

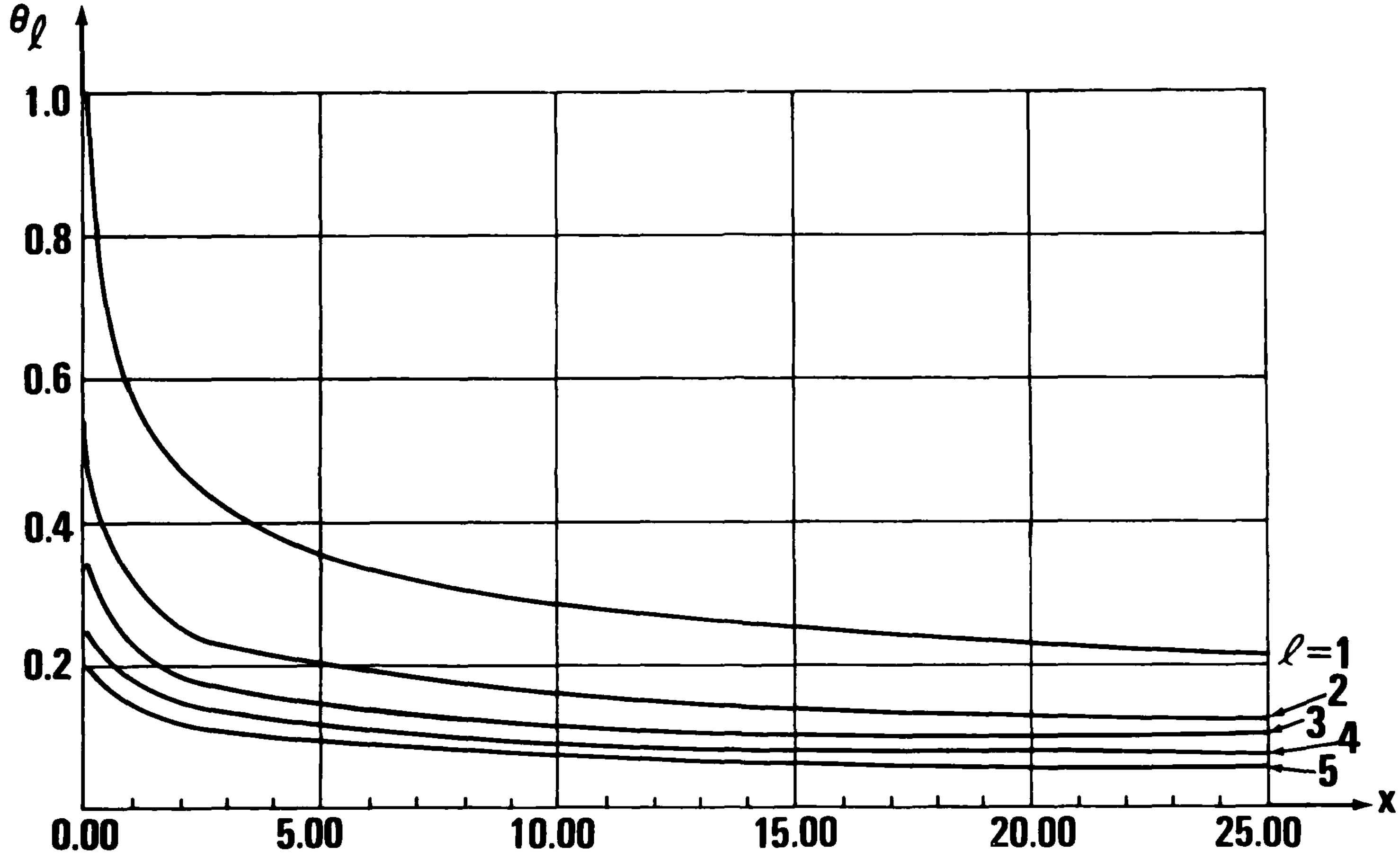

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Fig. 2b

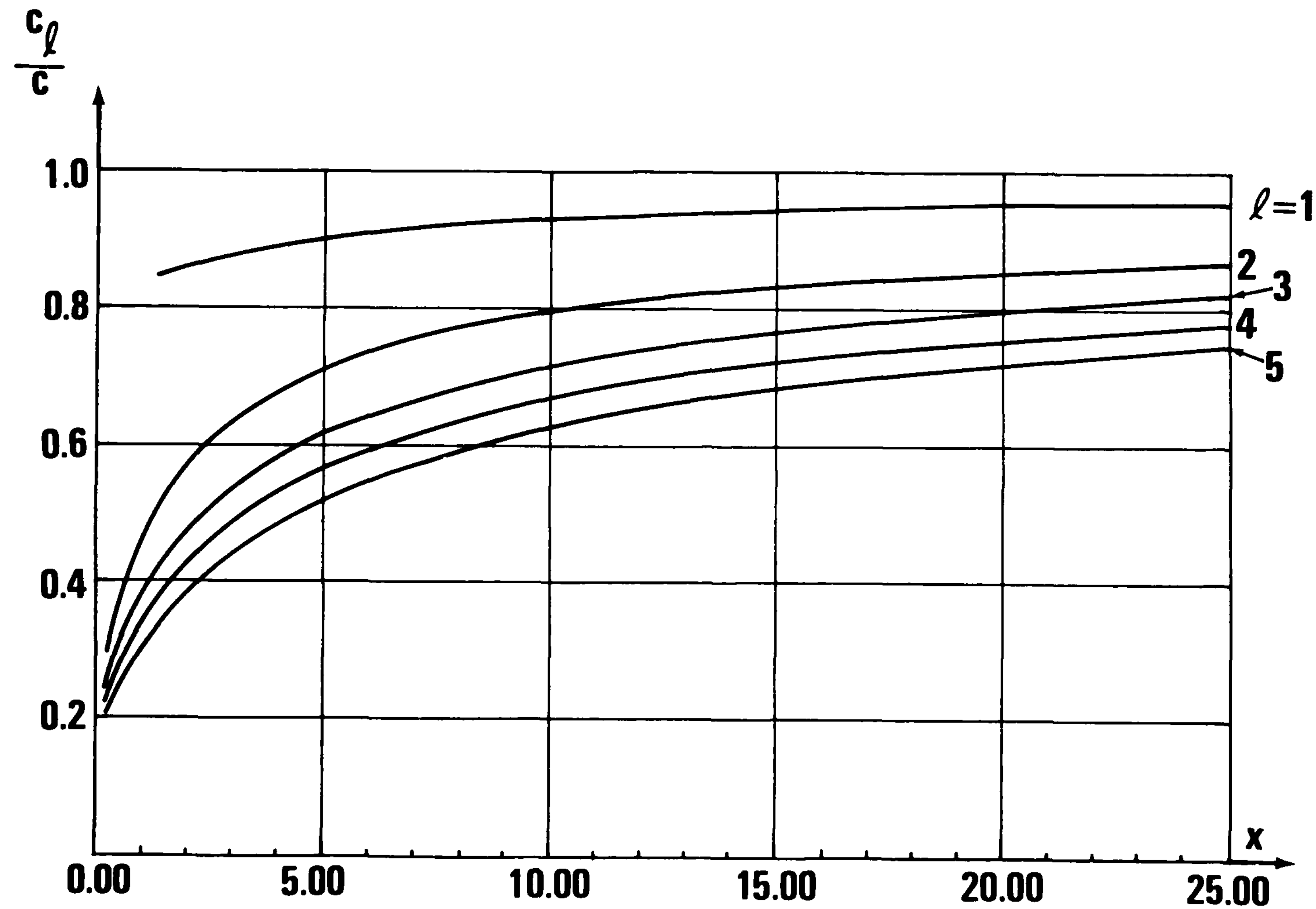

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Fig. 3a

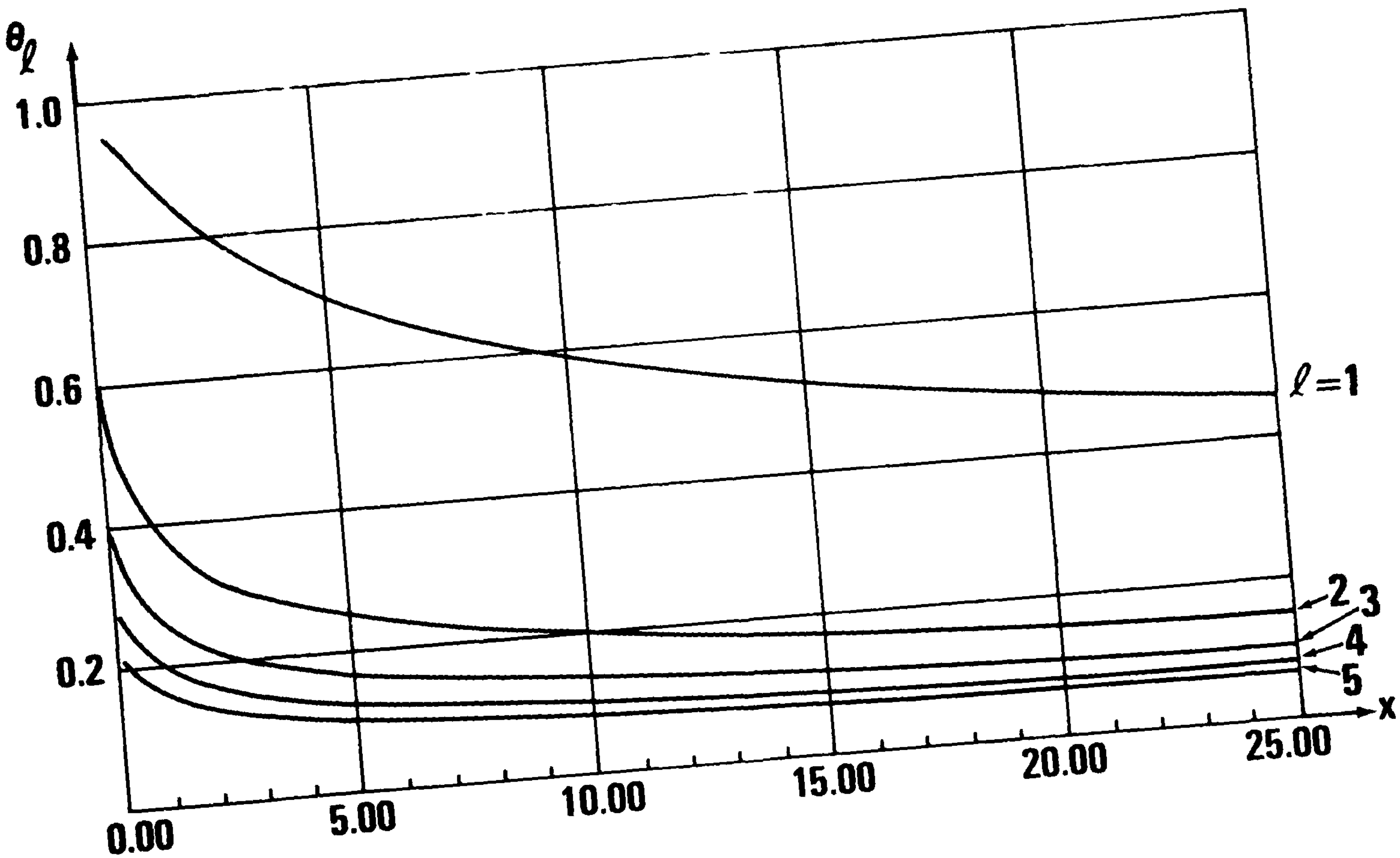

Fig. 3b

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INTERIOR AND EXTERIOR RESONANCES IN ACOUSTIC SCATTERING II: TARGETS OF ARBITRARY SHAPE (T-MATRIX APPROACH)

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The T-matrix approach, which describes the scattering of acoustic waves (or of other waves) from objects of arbitrary shape and geometry, is here "married" to the resonance scattering theory in order to obtain the (complex) resonance frequencies of an arbitrary-shaped target. For the case of nearly-impenetrable targets, we split the partial-wave scattering amplitudes into terms corresponding to internal resonances, plus an apparently non-resonant background amplitude which, however, contains the broad resonances caused by external diffracted (or Franz-type, creeping) waves, in addition to geometrically reflected and refracted (ray) contributions.

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INTRODUCTION

The preceding paper[1] has discussed, using the example of a spherical, near-impenetrable scatterer, how the scattering amplitude may be split into internal and external resonance contributions, in addition to geometrically reflected and refracted (ray) contributions which are obtained when the Watson transformation is utilized. It has furthermore shown how the external as well as the internal resonance amplitudes may be associated with external and internal surface waves, respectively. Quite the same demonstration is expected to be feasible for scatterers of arbitrary shape, at least if they are convex and of smooth curvature (while discontinuities and concavities will cause additional scattering effects, such as edge diffraction[2] and multiple scattering).

In the present paper, some parts of such a demonstration will be carried out. On the basis of Waterman's[3-6] "T-matrix approach", which permits the calculation of scattering amplitudes from targets objects of arbitrary shape, we separate the scattering amplitude into a part that corresponds to the internal resonances (or surface waves) of the penetrable target (assumed to be nearly impenetrable), and into a background that contains external resonances (or surface waves) of the target. Explicit eigenvalue equations for the resonance frequencies are written down, and the resonant scattering amplitudes are cast into the standard Breit-Wigner[7] resonance form

The T-matrix formalism is thus "married" here to the "Resonance Scattering Theory"[8], which was devised by us earlier for scatterers of separable geometry.

Alternately, using matrix diagonalizations, a formulation of the scattering amplitude in terms of resonant eigen-channels is obtained. We also demonstrate that a frequency average of the scattering cross section over the internal resonances results in the cross section of an impenetrable target of the same shape. This, as well as the resonance properties, provide powerful tools for the determination of target shape and composition from an analysis of the frequency dependence of the echo signal.

I. T-MATRIX FORMALISM FOR PENETRABLE AND IMPENETRABLE SCATTERERS.

The Waterman formalism[3-6] for calculating the amplitude or the cross section of wave scattering from irregularly-shaped objects proceeds via an expansion of the fields in terms of three-dimensional spherical basis functions. For the case of incident acoustic waves in an ambient fluid, and for a fluid scatterer as it will be assumed here, scalar basis functions are needed:

$$\psi_n(\vec{r}) \equiv \psi_{mn}^{\sigma}(\vec{r}) = \gamma_{mn}^{1/2} h_n^{(1)}(kr) Y_{mn}^{\sigma}(\hat{r}) \qquad (1a)$$

($n = 0,1,\ldots\infty$, $m = 0,1,\ldots n$, $\sigma = 1,2$), where

$$Y_{mn}^{\sigma}(\hat{r}) = P_n^m(\cos\theta)\begin{cases}\cos m\phi & (\sigma=1)\\ \sin m\phi & (\sigma=2)\end{cases} \tag{1b}$$

are the surface spherical harmonics, and

$$\gamma_{mn} = \varepsilon_m (2n+1)(n-m)! \,/\, [4\pi\,(n+m)!\,], \tag{1c}$$

where $\varepsilon_m = 1$ ($m = 0$), $\varepsilon_m = 2$ ($m \geq 1$). A time dependence $\exp(-i\omega t)$ is assumed, so that Eq. (1a) represents outgoing spherical waves. We also introduce standing waves

$$\hat{\psi}_n(\vec{r}) \equiv \hat{\psi}_{mn}^{\sigma}(\vec{r}) = \gamma_{mn}^{1/2}\, j_n(kr)\, Y_{mn}^{\sigma}(\hat{r}), \tag{1d}$$

where the caret denotes the regular part of ψ_{mn}^{σ} . In Eqs. (1a) and (1d), the first expression symbolizes the indices ($mn\sigma$) by the single symbol n.

The target object, of volume V and (arbitrarily-shaped) surface S, assumed homogeneous, is characterized by a density ρ_0 and sound speed c_0, or a propagation constant $k_0 = \omega/c_0$, while the ambient fluid has ρ , c, and k, respectively. We expand the incident wave $u^{(i)}(\vec{r})$ in terms of the basis of Eq. (1d),

$$u^{(i)}(\vec{r}) = \sum_n a_n \hat{\psi}_n(\vec{r}), \tag{2a}$$

which is regular at the origin. The expansion coefficients $a_n \equiv a_{mn}^{\sigma}$ are known, since they may be obtained from the known incident wave and Eq. (2a) via the orthogonality relations of $\hat{\psi}_{mn}^{\sigma}(\vec{r})$.

The total field $u(\vec{r})$ outside of the scattering object is given by

$$u(\vec{r}) = u^{(i)}(\vec{r}) + u^{(s)}(\vec{r}) \tag{2b}$$

since a scattered wave $u^{(s)}(\vec{r})$ appears due to the presence of the scatterer; it is expanded in terms of Eq. (1a):

$$u^{(s)}(\vec{r}) = \sum_n c_n \psi_n(\vec{r}). \tag{2c}$$

Inside the scatterer, a refracted wave $u^{(o)}(\vec{r})$ appears which we expand as

$$u^{(o)}(\vec{r}) = \sum_n f_n \hat{\psi}_n^{(o)}(\vec{r}); \tag{2d}$$

here, $\hat{\psi}_n^{(o)}(\vec{r})$ are the basis functions of Eq. (1d) in which $k \equiv \omega/c$ is replaced by $k_o \equiv \omega/c_o$.

The Waterman theory subsequently proceeds to express the unknown expansion coefficients c_n and f_n in terms of the known ones, a_n, by satisfying the appropriate boundary conditions on the boundary surface S. For a fluid obstacle imbedded in a fluid, these are

$$\rho u(S) = \rho_o u^{(o)}(S), \tag{3a}$$

$$\partial u(S)/\partial n = \partial u^{(o)}(S)/\partial n, \tag{3b}$$

if $u(\vec{r})$ represents the velocity potential. The resulting relation

$$c_i = T_{ij}\, a_j \tag{4a}$$

determines the expansion coefficients of the scattered wave via a so-called T-matrix, which is symmetric, and is given by

$$T = -Q^{-1}\hat{Q}. \tag{4b}$$

Here, the elements of the matrix Q are[6]

$$Q_{jm} = k \int_S \left(\frac{\rho_0}{\rho} \hat{\psi}_j^{(0)} \frac{\partial \psi_m}{\partial n} - \psi_m \frac{\partial \hat{\psi}_j^{(0)}}{\partial n} \right) dS \tag{4c}$$

while those of $\hat{Q}$ are given by Eq. (4c) with ψ_m replaced by $\hat{\psi}_m$. We may, in addition, introduce a matrix $\overline{Q}$, with elements also given by Eq. (4c) but with ψ_m replaced by $\overline{\psi}_m$, the latter being defined by Eq. (1a) with $h^{(2)}(kr)$ substituted for $h^{(1)}(k)$. This leads to

$$\hat{Q} = \tfrac{1}{2}\,(Q + \overline{Q}). \tag{4d}$$

In place of the T matrix, we may also consider the S matrix which is defined by $S = 1 + 2T$, and which thus becomes

$$S = -Q^{-1}\overline{Q}. \tag{4e}$$

It should be noted at this point that Eq. (4a), being a matrix equation, no longer links a given partial wave amplitude of the scattered wave, c_i, to the same corresponding partial wave amplitude a_i of the incident wave, as had been the case for targets of separable geometry[1,8]. Rather, one incident partial wave a_j couples to <u>all</u> scattered partial waves c_i, due to the arbitrary (non-separable) geometry of the scattering object. The implications of this on the scattering resonances will be discussed below.

We shall now consider the limiting cases of impenetrable soft(s) or rigid (r) scatterers, which correspond to the Dirichlet $\left[u\,(\underset{\sim}{S}) = 0\right]$ or the Neumann boundary condition $\left[\partial u\,(\underset{\sim}{S})/\partial n = 0\right]$, respectively. The T and S matrices for these two cases are obtained as

$$T^{(s,r)} = -Q^{(s,r)-1}\,\hat{Q}^{(s,r)}, \tag{5a}$$

$$S^{(s,r)} = -Q^{(s,r)-1}\,\bar{Q}^{(s,r)}, \tag{5b}$$

where[6]

$$Q^{(s)}_{jm} = -k\int_S \psi_m \frac{\partial\hat{\psi}_j}{\partial n}\,dS, \tag{5c}$$

$$Q^{(r)}_{jm} = k\int_S \hat{\psi}_j \frac{\partial\psi_m}{\partial n}\,dS, \tag{5d}$$

and where $\hat{Q}^{(s,r)}$ or $\bar{Q}^{(s,r)}$ are obtained from $Q^{(s,r)}$ as discussed after Eq. (4c).

II. INTERNAL VS. EXTERNAL RESONANCES

The matrices Q and T are of infinite dimensions, but in practice they will be taken as square matrices of finite dimension, corresponding to a truncation of the partial wave series. The inverse matrices Q^{-1} appearing in Eqs. (4) and (5) may then be written explicitly in the well-known way by

$$Q^{-1} = \frac{q}{D}, \tag{6a}$$

$$Q^{(t)-1} = \frac{q^{(t)}}{D_t}, \tag{6b}$$

where t designates s or r, as the case may be, where

$$D = \det Q, \tag{6c}$$

$$D_t = \det Q^{(t)} \tag{6d}$$

and where q, or $q^{(t)}$, is the transpose of a matrix whose (i,j)th elements are given by the minors of Q or of $Q^{(t)}$,

respectively, with sign factors $(-1)^{i+j}$. Accordingly, the T and S matrices become

$$T = -\frac{1}{D}\, q\, \hat{Q}, \tag{7a}$$

$$S = -\frac{1}{D}\, q\, \bar{Q}, \tag{7b}$$

and

$$T^{(t)} = -\frac{1}{D_t}\, q^{(t)}\, \hat{Q}^{(t)}, \tag{8a}$$

$$S^{(t)} = -\frac{1}{D_t}\, q^{(t)}\, \bar{Q}^{(t)}. \tag{8b}$$

From these expressions, it is obvious that the complex poles of the S or T matrix (i.e., the complex eigenfrequencies of the target) are obtained from the characteristic equation

$$D(k) = 0, \quad k=k_\ell \quad (\ell = 1,2,\ldots) \tag{9a}$$

for the case of penetrable object, and from

$$D_t(k) = 0, \quad k = \hat{k}_\ell^{(t)} \quad (\ell = 1,2,\ldots) \tag{9b}$$

for an impenetrable scatterer of the same shape. (We loosely refer here to the eigenvalues of the wavenumber as "eigenfrequencies"). Two points should now be noted namely:

(a) The eigenfrequencies k_ℓ can no longer be labeled by the index of the partial wave in which the corresponding resonances occur as had been the case for separable geometries [cf. Eq. (9) of Reference 1], since in the present case all partial waves couple together. Hence, in each scattered partial wave <u>all</u> the resonances of the target will appear simultaneously. (Of course, for a target whose shape is close to separable, one subset of resonances is expected to

dominate in a given partial wave and appear only weakly in the neighboring partial waves, thus approaching the separable situation). However, if the eigenchannel approach which is outlined further below is adopted, separate resonance frequencies may again be defined for each eigen-channel.

(b) The set of eigenfrequencies k_ℓ , Eq. (9a), will include both those corresponding to internal resonances, i.e. where most of the vibrational energy resides inside the target object (and with Im k_ℓ being small), and those corresponding to external resonances with most vibrational energy residing in the ambient fluid in the vicinity of the target (which have relatively large values of Im k_ℓ). For the case of a near-impenetrable target as will be assumed here, the external resonance frequencies will coincide closely with the solutions $\hat{k}_\ell^{(t)}$ of Eq. (9b), which for practical purposes will thus form a subset of k_ℓ .

Since we shall here be mostly interested in the internal resonances of the target, we introduce the quantity

$$d = D/D_t \tag{10a}$$

and note that for a target which is sufficiently close to impenetrable, the characteristic equation

$$d(k) = 0, \qquad k = \tilde{k}_\ell \quad (\ell = 1,2,\ldots) \tag{10b}$$

has for its complex roots only those corresponding to the <u>internal</u> resonances of the scatterer, since its external roots nearly coincide with those of D_t and thus effectively drop out from the ratio of Eq. (10a). Furthermore, since

internal resonance frequencies have small imaginary parts[1,8], the real internal resonance frequencies (defined as the real frequencies at which the scattering amplitude exhibits resonance features), which we denote by $k_\ell^{(t)}$ for a nearly-impenetrable target t = s or r, are approximately given by

$$k_\ell^{(t)} = \mathrm{Re}\,\tilde{k}_\ell\ , \tag{10c}$$

and are approximately obtained as the solutions of the real "characteristic equation"

$$\mathrm{Re}\ d\ (k) \equiv \mathrm{Re}\ \frac{\mathcal{D}(k)}{\mathcal{D}_t(k)} = 0, \qquad k = k_\ell^{(t)} \quad (\ell = 1, 2, \ldots). \tag{10d}$$

If now the T or S matrix of Eqs. (7) is written as

$$T = -\ \frac{q\,\hat{Q}}{\mathcal{D}_t\, d}\ , \tag{11a}$$

$$S = -\ \frac{q\,\bar{Q}}{\mathcal{D}_t\, d}\ , \tag{11b}$$

we see that the internal poles (complex roots of d) have been separated from the external poles (complex roots of D_t) of these matrices in a fashion which is approximately applicable to nearly-impenetrable targets.

III. SEPARATION OF INTERNAL RESONANCES AND BACKGROUND

By its definition[9], the S matrix connects the spherically ingoing partial wave amplitudes of the incident wave with the spherically outgoing (incident and scattered) partial waves. The T matrix, however, governs the amplitudes of the scattered partial waves only. While thus the S matrix elements contains poles corresponding to internal as well as external resonances, the T matrix elements can be shown to separate into terms containing both types of resonances, plus background terms containing the (broad) external resonances only. This was demonstrated in detail for targets of separable geometry[8], and will be shown here to hold for targets of arbitrary shape also. In our formulation, we shall closely follow that of the separable-geometry case[8].

We have for the T-matrix generally

$$T = \tfrac{1}{2}\,(S - 1). \tag{12a}$$

Using the form of Eq. (11b) for S, and substituting for $-1/D_t$ the expression

$$-1/D_t = S^{(t)}\bar{Q}^{(t)-1}\,\mathcal{Q}^{(t)-1} \tag{12b}$$

which follows form Eq. (8b), we obtain

$$T = \tfrac{1}{2}\,S^{(t)}\left(\frac{1}{d}\,\bar{Q}^{(t)-1}\mathcal{Q}^{(t)-1}\,\mathcal{Q}\bar{Q} - S^{(t)-1}\right). \tag{12c}$$

Since S is unitary[5], i.e. $S^{-1} = S^{\dagger}$ where the dagger denotes the Hermitean conjugate, we have

$$S^{(t)-1} = S^{(t)\dagger} = 1 + 2T^{(t)\dagger}. \tag{12d}$$

Inserting this in Eq. (12c) results in the desired form for the T matrix:

$$T = S^{(t)} \left(\tfrac{1}{2} X - T^{(t)\dagger} \right) \tag{13a}$$

where

$$X = \frac{1}{d} \bar{Q}^{(t)-1} q^{(t)-1} q \bar{Q} - 1 . \tag{13b}$$

Expressing

$$d = D/D_t = Q^{(t)-1} q^{(t)-1} q Q , \tag{13c}$$

we obtain our final expression

$$X = \frac{1}{d} \left(\bar{Q}^{(t)-1} q^{(t)-1} q \bar{Q} - Q^{(t)-1} q^{(t)-1} q Q \right), \tag{13d}$$

to be used in Eq. (13a). These results are exactly analogous to the separable case. For a nearly-rigid cylinder or sphere, Eq. (23a) of Reference 8 gives

$$T_n = S_n^{(r)} \left\{ \frac{1/Z_n^{(1)} - 1/Z_n^{(2)}}{1/F_n - 1/Z_n^{(1)}} - T_n^{(r)\dagger} \right\}, \tag{14a}$$

which is analogous to Eq. (13a), and which also agrees with Eq. (7b) of Reference 1 since

$$-S_n^{(t)} T_n^{(t)\dagger} = T_n^{(t)} , \tag{14b}$$

or generally

$$- ST^{\dagger} = T \tag{14c}$$

for S,T and $S^{(t)}$,$T^{(t)}$ matrices. For separable geometry, the matrix equations Eqs. (13a) and (13d) reduce to their diagonal terms, for the nearly-rigid case reproducing exactly Eq. (14a) since here,

$$\tfrac{1}{2} X \rightarrow \frac{1/z_n^{(1)} - 1/z_n^{(2)}}{1/F_n - 1/z_n^{(1)}} . \tag{14d}$$

We may also rewrite

$$T = T^{(t)} + T^{(t)\,int} \tag{15a}$$

to make it analogous to Eq. (7b) of Reference 1, with

$$T^{(t)\,int} = \tfrac{1}{2}\, S^{(t)} X \equiv \tfrac{1}{2}\, S^{(t)\,int} . \tag{15b}$$

The T matrix is thus split into a background portion corresponding to scattering from an impenetrable target of the same shape, $T^{(t)}$, plus a portion $T^{(t)\,int}$ that contains the internal target resonances due to the factor 1/d of the matrix X. As in Reference 1, the background portion still contains the external resonances due to the Franz-type creeping waves, corresponding to the poles of $S^{(t)}$.

IV RESONANCE FORMULATION

The denominator $d \equiv D/D_t$ of the X matrix, Eq. (13d), may be split into real and imaginary parts,

$$d(k) = d_R(k) + i\, d_I\,(k), \tag{16a}$$

and the resonance frequencies $k_\ell^{(t)}$ are obtained as the real roots of $d_R(k) = 0$, Eq. (10d). We perform a Taylor expansion about these resonance frequencies,

$$d(k) \cong (k - k_\ell^{(t)})\, d_R'(k_\ell^{(t)}) + i d_I(k_\ell^{(t)}), \tag{16b}$$

and introduce the resonance widths

$$\Gamma_\ell^{(t)} = 2\, \frac{d_I(k_\ell^{(t)})}{d_R'(k_\ell^{(t)})}, \tag{16c}$$

which corresponds to narrow internal resonances since here both d_I and the imaginary parts of the complex eigenfrequencies $\tilde{k}_\ell$, Eq. (10b), are small. Expressing

$$X = \frac{1}{d} Y \tag{17a}$$

where

$$Y = \bar{Q}^{(t)-1} q^{(t)-1} q \bar{Q} - Q^{(t)-1} q^{(t)-1} q Q, \tag{17b}$$

Eq. (16) permit us to approximate

$$X(k) \cong \frac{1}{d_R(k_\ell^{(t)})} \frac{Y(k_\ell^{(t)})}{k - k_\ell^{(t)} + \frac{i}{2}\Gamma_\ell^{(t)}} \tag{17c}$$

in the vicinity of each resonance frequency $k_\ell^{(t)}$. The T matrix of Eq. (13a) may thus be written as

$$T = T^{(t)} + T^{(t)int} \tag{18a}$$

where $T^{(t)int}$ now is in the resonance (Breit-Wigner[7]) form,

$$T^{(t)int} \cong \frac{1}{2} \sum_\ell S^{(t)}(k_\ell^{(t)}) \frac{Y(k_\ell^{(t)})}{d_R'(k_\ell^{(t)})} \frac{1}{k - k_\ell^{(t)} + \frac{i}{2}\Gamma_\ell^{(t)}}. \tag{18b}$$

The internal resonances amplitudes in the T matrix may thus be obtained, for the case of a nearly-impenetrable scatterer, by subtracting element-by-element the corresponding T-matrix

$T^{(t)}$ of the impenetrable scatterer of same shape from the total T matrix.

V. EIGENCHANNEL FORMULATION

It was stated after Eq. (9) that due to channel coupling, all resonances of the target will appear in all scattered partial waves channels so that the partial wave number is no longer a valid label for the resonances. Matrix diagonalization will, however, introduce a new set of uncoupled "eigenchannels" of the target's S matrix, each of which having its own specific set of resonances.

We start from Eqs. (4b), (4e), (5a) and (5b) and diagonalize the (truncated nxn) matrices Q and $Q^{(t)}$ appearing therein:

$$UQU^{-1} = \Lambda, \tag{19a}$$

$$U^{(t)}Q^{(t)}U^{(t)-1} = \Lambda^{(t)} \tag{19b}$$

where Λ is diagonal with elements λ_i, and $\Lambda^{(t)}$ is diagonal with elements $\lambda_i^{(t)}$; further, Λ^{-1} is diagonal with elements $1/\lambda_i$, and

$$\det \Lambda = \lambda_1 \lambda_2 \ldots \lambda_n . \tag{19c}$$

We have

$$Q^{-1} = U^{-1} \Lambda^{-1} U, \tag{20a}$$

$$Q^{(t)-1} = U^{(t)-1} \Lambda^{(t)-1} U^{(t)}. \tag{20b}$$

If we now designate the matrices

$$U^{-1} = u, \quad U\hat{Q} = \hat{v}, \quad U\bar{Q} = \bar{v}, \tag{20c}$$

$$U^{(t)-1} = u^{(t)}, \quad U^{(t)}\hat{Q}^{(t)} = \hat{v}^{(t)}, \quad U^{(t)}\bar{Q}^{(t)} = \bar{v}^{(t)}, \tag{20d}$$

then we find from Eqs. (4b), (4e), (5a) and (5b):

$$T = -u\Lambda^{-1}\hat{v}, \tag{21a}$$

$$S = -u\Lambda^{-1}\bar{v}, \tag{21b}$$

$$T^{(t)} = -u^{(t)}\Lambda^{(t)-1}\hat{v}^{(t)}, \tag{21c}$$

$$S^{(t)} = -u^{(t)}\Lambda^{(t)-1}\bar{v}^{(t)}. \tag{21d}$$

Making use of the diagonal nature of Λ^{-1} and $\Lambda^{(t)-1}$, we find for the individual T and S matrix elements:

$$T_{ij} = -\sum_m \frac{u_{im}\hat{v}_{mj}}{\lambda_m}, \tag{22a}$$

$$S_{ij} = -\sum_m \frac{u_{im}\bar{v}_{mj}}{\lambda_m}, \tag{22b}$$

$$T_{ij}^{(t)} = -\sum_m \frac{u^{(t)}_{im}\hat{v}_{mj}^{(t)}}{\lambda_m^{(t)}}, \tag{22c}$$

$$S_{ij}^{(t)} = -\sum_m \frac{u^{(t)}_{im}\bar{v}_{mj}^{(t)}}{\lambda_m^{(t)}}. \tag{22d}$$

These expressions represent what may be termed an "eigen-channel" decomposition of the T and S matrix elements. The mth eigenchannel amplitude has a resonance denominator λ_m (or $\lambda_m^{(t)}$), and the complex eigenfrequencies are obtained from the characteristic equations

$$\lambda_m(k) = 0, \quad k = k_{m\ell} \; (\ell = 1,2,\dots) \tag{23a}$$

corresponding to both internal and external resonances. The external subset of eigenfrequencies is, for nearly-impenetrable objects, obtained approximately from the characteristic equations for impenetrable targets,

$$\lambda_m^{(t)}(k)=0, \qquad k=\hat{k}_{m\ell}^{(t)} \qquad (\ell=1,2,\ldots). \tag{23b}$$

Note that now, the eigenfrequencies are characterized by the label m of the mth eigenchannel. This implies that each eigenchannel has its own specific set of eigenfrequencies, different from the eigenfrequencies of the other eigenchannels. The totality of eigenfrequencies $k_{m\ell}$, however, is the same as that of the set k_ℓ, Eq. (9a), and the totality of $\hat{k}_{m\ell}^{(t)}$ is the same as the set $\hat{k}_\ell^{(t)}$, Eq. (9b), simply because the unitary transformations of Eqs. (19) leave the determinants invariant:

$$D = \det Q = \det \Lambda = \lambda_1 \lambda_2 \cdots \lambda_n, \tag{24a}$$

$$D_t = \det Q^{(t)} = \det \Lambda^{(t)} = \lambda_1^{(t)} \lambda_2^{(t)} \cdots \lambda_n^{(t)}. \tag{24b}$$

By use of the eigenchannel theory, the eigenvalues corresponding to internal as well as external resonances have thus been reclassified into different subsets, each subset corresponding to a different eigenchannel m.

In the resonance theory, the eigenchannel denominators λ_m may again, as in Eq. (16b), be Taylor-expanded about the resonance frequencies corresponding to the internal resonances. For such a purpose, Eqs. (22) constitute an automatic partial-fraction decomposition of the resonance denominators. A background subtraction separating internal from external resonance amplitudes may again be carried out as in Section III, but this time in the framework of the eigenchannel theory; this is left as an exercise for the reader.

VI. FREQUENCY AVERAGE OVER RESONANCES

As in the preceding paper[1], we shall show here that the (narrow) internal resonances disappear from the scattering cross section when frequency-averaged over their width.

From Eqs. (2) and (4), the differential cross section is obtained as

$$\frac{d\sigma}{d\Omega} = \frac{1}{k^2}\left|\sum_{\substack{mn\sigma \\ m'n'\sigma'}} (-i)^n \gamma_{mn}^{1/2}\, Y_{mn}^{\sigma}(\hat{r})\, T_{mn\sigma,\, m'n'\sigma'}\, a_{m'n'}^{\sigma'}\right|^2. \tag{25a}$$

If we now insert the resonance expressions Eqs. (18) for T, and average over frequencies, it is easy to see just as in Section VII of Reference 1, that the frequency average eliminates the cross terms and square terms containing $T^{(t)\,int}$, so that we are left with

$$\left\langle\frac{d\sigma}{d\Omega}\right\rangle = \frac{1}{k^2}\left|\sum_{\substack{mn\sigma \\ m'n'\sigma'}} (-i)^n \gamma_{mn}^{1/2}\, Y_{mn}^{\sigma}(\hat{r})\, T^{(t)}_{mn\sigma,\, m'n'\sigma'}\, a_{m'n'}^{\sigma'}\right|^2. \tag{25b}$$

This is, however, the cross section from an impenetrable scatterer of the same shape as the penetrable object under consideration[10]. The foregoing observation shows that we now have two powerful tools at our disposal in order to identify a given target from its echo spectrum (or form function), namely:

(a) the form function in which the internal resonances have been frequency-averaged out, as a determinator of shape

(b) the resonances in the form function, as determinators of both target shape and internal composition of the target.

VI. CONCLUSION

The present study has extended a number of the concepts, discussed in the preceding paper[1] for targets of separable geometry, to the case of targets of arbitrary geometry using the T matrix approach of Waterman. In particular, the resonance theory previously developed for targets of separable geometry[8] has been "married" to the T matrix theory of scattering from objects of general shapes, demonstrating (for the case of nearly-impenetrable targets) that an analytic separation of resonant amplitudes corresponding to the internal eigenvibrations of the target from a background containing only external resonances may be carried out in the framework of such an approach. Characteristic T matrix equations for the complex eigenfrequencies and the real resonance frequencies of the target have been obtained, and the sets of eigenfrequencies have been shown to be classifiable in terms of "eigenchannels" of the diagonalized scattering problem, i.e. according to the eigenchannel in which each set of eigenvibrations appears.

It would be straight-forward to extend the geometrical and physical interpretations of the resonances, as carried out for separable goemetrics in the preceding paper[1], to the present nonseparable case via a use of the Watson transformation. In particular, the internal and external resonances would then be shown to be caused by the phase matching of internal and external circumferential waves, respectively, which propagate along geodesic paths over the surface of the target,

and the resonance frequencies would thus be linked to the phase velocities of these waves so that the dispersion curves of the latter could be obtained for each given geodesic. Finally, geometrically reflected and refracted contributions could also shown to be present from the use of the Watson transformation.

The present and preceding[1] studies, although restricted to nearly-impenetrable scatterers (which, however, are of great practical importance), are deemed to provide us with a comprehensive picture of the scattering phenomena, both resonant and nonresonant, from penetrable objects. This picture should be applicable with only minor changes of formalism, to acoustic scattering, and to the scattering of elastic or electromagnetic waves. The fact that scatterers of arbitrary geometry have been included in the discussion, points to a practical use of the phenomena described here (in particular, of the resonances) for a classification of targets according to both their shape and their material composition.

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